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Data analysis and entropy steered discrete filtering for the numerical treatment of conservation laws

Thorsten Grahs*,† and Thomas Sonar‡

Institute for Analysis, Technical University of Braunschweig, Pockelsstr. 14, D-38106 Braunschweig, Germany

SUMMARY

This paper describes the construction of a discrete filter algorithm for second-order oscillatory schemes applied to scalar conservation laws in two space dimensions. Starting from a modification of the Lax–Wendroff scheme proposed by LeVeque (Numer. Meth. Fluids 1993; **4**:175; SIAM J. Numer. Anal. 1996; **33**(2):627) we extend this algorithm with an anisotropic diffusion procedure in order to smooth the spurious oscillations in the vicinity of a shock. To locate the regions identified with the discontinuity we analyse the discrete data with the help of an indicator for entropy production origin from a discrete entropy inequality. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: discrete filter; anisotropic diffusion

1. GOVERNING EQUATIONS

We consider a scalar hyperbolic conservation law in two dimensions

$$\partial_t u + \partial_x f(u) + \partial_y g(u) = 0 \tag{1}$$

where the fluxes f and g are assumed to be differentiable. Independently of the smoothness of the initial conditions discontinuities develop in general within a finite time so that weak solutions u defined by

$$\int_\omega (u\partial_t \Phi + f(u)\partial_x \Phi + g(u)\partial_y \Phi)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = 0$$

have to be considered. Here, $\omega \subset \mathbb{R}_0^+ \times \mathbb{R}^2$, $\Phi \in C_0^1(\omega)$ and $u \in L^1 \cap L^\infty(\omega)$.

Weak solutions are not uniquely defined and an entropy condition is needed to single out the physically relevant solution. (1) is augmented with the entropy inequality

$$\partial_t \eta(u) + \partial_x \psi(u) + \partial_y \varphi(u) \leqslant 0 \tag{2}$$

*Correspondence to: T. Grahs, Institute for Analysis, Technical University of Braunschweig, Pockelsstr. 14, D-38106 Braunschweig, Germany.
†E-mail: t.grahs@tu-bs.de
‡E-mail: t.sonar@tu-bs.de

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holding for all triples (η, ψ, ϕ), where η is a convex function and ψ and ϕ are entropy fluxes compatible in the following sense (see Reference [1] for details):

$$\psi(u) = \int_0^u \eta'(\xi) f'(\xi)\,\mathrm{d}\xi, \quad \varphi(u) = \int_0^u \eta'(\xi) g'(\xi)\,\mathrm{d}\xi \tag{3}$$

2. DATA ANALYSIS

As a discrete analogue of (2) we introduce the discrete entropy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta_{i,j}(t) + \frac{1}{\Delta x}(\Psi^n_{i+1/2,j} - \Psi^n_{i-1/2,j}) - \frac{1}{\Delta y}(\Phi^n_{i,j+1/2} - \Phi^n_{i,j-1/2}) \leqslant 0 \tag{4}$$

where $\eta_{i,j}(t)$ is the entropy located at a grid point and $\Psi^n_{i+1/2,j}$ and $\Phi^n_{i,j+1/2}$ are numerical entropy fluxes consistent with ψ and φ in just the same sense as the numerical fluxes are with f and g, see Reference [2].

In principle, one is free to choose arbitrary numerical entropy fluxes as long as they are consistent with the original ones. However, it was shown in Reference [3] that certain pathologies may occur if the discrete numerical entropy is treated differently from the numerical fluxes. In our understanding every oscillation occurring in the numerical solution should violate the discrete entropy inequality since there are jumps 'in the wrong direction' included. If the numerical entropy fluxes are chosen without reference to the discretization of the numerical fluxes schemes exhibiting hefty oscillations may satisfy a discrete entropy inequality, see Reference [4]. Hence, we propose Lax-c consistent numerical fluxes as outlined in Reference [3]. If

$$H^n_{i+1/2} = \tfrac{1}{2}(f^n_{i+1} + f^n_i) - Q^n_{i+1/2,j}(U^n_{i+1,j} - U^n_{i,j})$$

is the numerical flux for f, then Ψ should be discretized as

$$\Psi^n_{i+1/2} = \tfrac{1}{2}(\psi^n_{i+1} + \psi^n_i) - \mathcal{Q}^n_{i+1/2,j}(\eta^n_{i+1,j} - \eta^n_{i,j})$$

The numerical entropy dissipation $\mathcal{Q}$ is derived from Q by replacing $U^n_{i,j}$ by $\eta^n_{i,j}$ and replacing $f^n_{i,j}$ by $\psi^n_{i,j}$.

As already mentioned conservation laws obey an entropy equality. In smooth regions even an entropy equality is fulfilled, i.e.

$$\partial_t \eta(u) + \partial_x \psi(u) + \partial_y \varphi(u) = 0$$

If we look at the discretize model of the entropy inequality, we have

$$\eta^{n+1} = \eta^n - \lambda_x(\Psi^n_{i+1/2,j} - \Psi^n_{i-1/2,j}) - \lambda_y(\Phi^n_{i,j+1/2} - \Phi^n_{i,j-1/2})$$

In the vicinity of discontinuities in the numerical solution for high-order methods, normally a violation of this numerical entropy conditions will occur due to the oscillations at the shock front. These both tasks are related, but in a highly non-trivial way.

So the unmodified Roe scheme [5] possesses the TVD property, but the limit solution allows entropy violating shocks. On the other hand, the Lax–Wendroff fix by Majda and Osher [6] produces still oscillations, but satisfies a discrete entropy inequality.

Hence one sees that this is a difficult task, but nevertheless we assume regions where the discrete entropy inequality is violated as region were one has to consider additional dissipation.

So we define a so-called *entropy production*

$$e_{i,j}^{+} := \Delta\eta_{i,j}^{n+1,+} := \begin{cases} \eta_{i,j}^{n+1} - \eta_{i,j}^{n}, & \eta_{i,j}^{n+1} > \eta_{i,j}^{n} \\ 0, & \text{else} \end{cases} \tag{5}$$

We use this indicator in the following to distinct between regions where the solution is smooth and regions with entropy production, where we have to add additional numerical dissipation.

We remark, that maybe at rarefaction waves the indicator will add diffusion, but due to the smoothness of the solution in this region this will be neglectable.

3. THE BASIC SCHEME

We consider conservative finite difference schemes as discrete models of conservation laws, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}U_{i,j}(t) = -\frac{1}{\Delta x}(F_{i+1/2,j}^{n} - F_{i-1/2,j}^{n}) - \frac{1}{\Delta y}(G_{i,j+1/2}^{n} - G_{i,j-1/2}^{n}) \tag{6}$$

where $F_{i\pm 1/2,j}^{n}$ and $G_{i,j\pm 1/2}^{n}$ denote numerical flux functions consistent with f and g in the sense of $F(\dots,u,u,\dots) = f(u)$, $G(\dots,u,u,\dots) = g(u)$. As the basic scheme we use the Lax–Wendroff scheme with the modification proposed by LeVeque [7, 8]. This scheme uses an additional discretization of the cross fluxes where $A = f'$, $B = g'$:

$$\begin{aligned} ABu_y &\approx \frac{1}{4h}AB(\Delta_y U_{i-1,j} + \Delta_x U_{i-1,j+1} + \Delta_x U_{i,j} + \Delta_y U_{i,j+1}) \\ &\approx \frac{1}{4h}(A^+B^+\Delta_y U_{i-1,j} + A^+B^-\Delta_x U_{i-1,j+1} + A^-B^+\Delta_x U_{i,j} + A^-B^-\Delta_y U_{i,j+1}) \end{aligned} \tag{7}$$

Here A^+ (resp. A^-) represents a correction wave from the left (resp. right) while B^+ (resp. B^-) represents information travelling upward (resp. downward) into the corresponding cell. So the numerical fluxes read as follows:

$$\begin{aligned} F_{i+1/2,j} &:= \hat{F}_{i+1/2,j} + \tfrac{1}{2}\lambda\tilde{F}_{i+1/2,j} \\ \tilde{F}_{i+1/2,j} &:= -(A^-B^-)_{i+1,j+1/2}\Delta_y u_{i+1,j+1} - (A^+B^-)_{i,j+1/2}\Delta_y u_{i,j+1} \\ &\quad -(A^-B^+)_{i+1,j-1/2}\Delta_y u_{i+1,j} - (A^+B^+)_{i,j-1/2}\Delta_y u_{i,j} \end{aligned}$$

where $\hat{F}_{i+1/2,j}$ is the usual Lax–Wendroff flux for the right-cell face. The fluxes $F_{i-1/2,j}$, $G_{i,j+1/2}$, $G_{i,j-1/2}$ write in an adequate manner.

If we look at the differences of the numerical flux functions, we can interprete the discretization of the cross derivatives as an anisotropic diffusion with non-linear diffusion coefficients A^+, A^-, B^+, B^-:

$$\tilde{F}_{i+1/2,j} - \tilde{F}_{i-1/2,j} \approx \partial_x(A^- B^- \partial_y U)_{i+1/2,j+1/2} + \partial_x(A^- B^+ \partial_y U)_{i+1/2,j-1/2} + \partial_x(A^+ B^- \partial_y U)_{i-1/2,j+1/2} + \partial_x(A^+ B^+ \partial_y U)_{i-1/2,j-1/2}$$

$$\tilde{G}_{i,j+1/2} - \tilde{G}_{i,j-1/2} \approx \partial_y(A^- B^- \partial_x U)_{i+1/2,j+1/2} \partial_y(A^- B^+ \partial_x U)_{i+1/2,j-1/2} + \partial_y(A^+ B^- \partial_x U)_{i-1/2,j+1/2} \partial_y(A^+ B^+ \partial_x U)_{i-1/2,j-1/2}$$

This is similar to a class of discrete filter algorithms from image processing (see References [9, 10]), we have already interpreted as anisotropic diffusion filter in the context of numerical treatment of conservation laws [11, 12].

So the question arises how to modify these correction to get an oscillation-free algorithm. As already mentioned we use the discrete entropy inequality as an indicator, letting us know in which regions we have to adjust the scheme. A scheme where we use the entropy production (5) directly to construct a filter can be found in Reference [12].

4. THE DISCRETE FILTER

If we are going to write the discretization (6) in the form

$$U_{i,j}^{n+1} = \sum_{l,k=-1}^{1} c_{lk} U_{i+l,j+k} \tag{8}$$

the original Lax–Wendroff scheme posses the following coefficients:

$$\begin{aligned} c_{i+1,j}&: \quad \tfrac{1}{2}\lambda[\lambda A_{i+1/2,j}^2 - A_{i+1/2,j}] \\ c_{i-1,j}&: \quad \tfrac{1}{2}\lambda[\lambda A_{i-1/2,j}^2 - A_{i-1/2,j}] \\ c_{i,j+1}&: \quad \tfrac{1}{2}\lambda[\lambda B_{i,j+1/2}^2 - B_{i,j+1/2}] \\ c_{i,j-1}&: \quad \tfrac{1}{2}\lambda[\lambda B_{i,j-1/2}^2 - B_{i,j-1/2}] \\ c_{i,j}&: \quad 1 - c_{i+1,j} - c_{i-1,j} - c_{i,j+1} - c_{i,j-1} \end{aligned}$$

If we are taking the discretization of the cross derivates (7) into account, they are modified in the following way:

$$\begin{aligned} \tilde{c}_{i,j} &:= c_{i,j} + A^-(B^- - B^+)_{i+1/2,j} + A^+(B^+ - B^-)_{i-1/2,j} \\ &\quad + B^-(A^- - A^+)_{i,j+1/2} + B^+(A^+ - A^-)_{i,j-1/2} \\ \tilde{c}_{i+1,j} &:= c_{i+1,j} + A^-(B^+ - B^-)_{i+1/2,j} - A^- B_{i+1,j+1/2}^- + A^- B_{i+1,j-1/2}^+ \end{aligned}$$

$$\tilde{c}_{i-1,j} := c_{i-1,j} + A^-(B^+ - B^-)_{i-1/2,j} - A^+B^+_{i-1,j-1/2} + A^+B^-_{i-1,j+1/2}$$

$$\tilde{c}_{i,j+1} := c_{i,j+1} + B^-(A^+ - A^-)_{i,j+1/2} - A^-B^-_{i+1/2,j+1} + A^+B^-_{i-1/2,j+1}$$

$$\tilde{c}_{i,j-1} := c_{i,j-1} + B^+(A^- - A^+)_{i,j-1/2} - A^+B^+_{i-1/2,j-1} + A^-B^+_{i+1/2,j-1}$$

$$\tilde{c}_{i+1,j+1} := c_{i+1,j+1} + A^-B^-_{i+1,j+1/2} + A^-B^-_{i+1/2,j+1}$$

$$\tilde{c}_{i-1,j+1} := c_{i-1,j+1} - A^+B^-_{i-1,j+1/2} - A^+B^-_{i-1/2,j+1}$$

$$\tilde{c}_{i+1,j-1} := c_{i+1,j-1} - A^-B^+_{i+1,j-1/2} - A^-B^+_{i-1/2,j-1}$$

$$\tilde{c}_{i-1,j-1} := c_{i-1,j-1} + A^+B^+_{i-1,j+1/2} + A^+B^+_{i-1/2,j-1}$$

Thus, the observation is that the correction waves alter the coefficients of the original scheme in the following way:

$$\tilde{c}_{i,j} \geqslant c_{i,j}$$

$$\tilde{c}_{i+1,j} \leqslant c_{i+1,j}, \qquad \tilde{c}_{i+1,j+1} \geqslant 0$$

$$\tilde{c}_{i-1,j} \leqslant c_{i-1,j}, \qquad \tilde{c}_{i+1,j-1} \geqslant 0$$

$$\tilde{c}_{i,j+1} \leqslant c_{i,j+1}, \qquad \tilde{c}_{i-1,j+1} \geqslant 0$$

$$\tilde{c}_{i,j-1} \leqslant c_{i,j-1}, \qquad \tilde{c}_{i-1,j-1} \geqslant 0$$

It is a well-known fact that a scheme written in form (8) is monotone if all coefficients $c_{k,l}$ are positive. With the help of an anisotropic diffusion algorithm we try to correct the scheme in order to remove the spurious oscillations.

The diffusion filter is constructed in the following way: if the indicator (5) is different from nil, i.e. indicates a region with entropy violating oscillations, we compute the sum of the positive coefficients,

$$\mathscr{C}^+ = \max(0.0, \tilde{c}_{i+1,j+1}) + \max(0.0, \tilde{c}_{i+1,j-1}) + \max(0.0, \tilde{c}_{i-1,j+1}) + \max(0.0, \tilde{c}_{i-1,j-1})$$

and the sum of the coefficients which are negative,

$$\mathscr{C}^- = |\min(0.0, \tilde{c}_{i,j}) + \min(0.0, \tilde{c}_{i+1,j}) + \min(0.0, \tilde{c}_{i-1,j}) + \min(0.0, \tilde{c}_{i,j+1}) + \min(0.0, \tilde{c}_{i,j-1})|$$

The amount that we can distribute is characterized by the ratio that we want to distribute, i.e. $\mathscr{C}^-$, and that we can distribute, i.e. $\mathscr{C}^+$. Since we do not want to create new negative weights, we have to limit this ratio by unity. So the distribution ratio reads

$$\mathscr{D} := \min(1, \mathscr{C}^-/\mathscr{C}^+)$$

Thus the diffusion coefficients for the cross diffusion weights are

$$a_{i\pm1/2,j\pm1/2}:=\tilde{c}_{i\pm1,j\pm1}\mathscr{D} \tag{9}$$

The diffusion aligned with the main axes depends on the amount the central coefficient gains from the corners. Thus we get

$$\mathscr{C}^{+}_{i,j}=\sum_{l,k=-1/2}^{1/2} a_{i+k,j+l}+\tilde{c}_{i,j}$$

and the sum of the coefficients which are negative,

$$\mathscr{C}^{-}_{i,j}=|\min(0.0,\tilde{c}_{i+1,j})+\min(0.0,\tilde{c}_{i-1,j})+\min(0.0,\tilde{c}_{i,j+1})+\min(0.0,\tilde{c}_{i,j-1})|$$

The corresponding weights for the diffusion correction read

$$a_{i\pm1/2,j}:=|\min(0.0,\tilde{c}_{i\pm1,j})|\mathscr{D}_{i,j},\quad a_{i,j\pm1/2}:=|\min(0.0,\tilde{c}_{i,j\pm1})|\mathscr{D}_{i,j} \tag{10}$$

With the coefficients (9,10) we can write the resulting scheme as

$$\begin{aligned}U^{n+1}_{i,j}=\hat{U}_{i,j}&+\tfrac{1}{2}\lambda[a_{i+1/2,j}(U^n_{i+1,j}-U^n_{i,j})-a_{i-1/2,j}(U^n_{i,j}-U^n_{i-1,j})\\&+a_{i,j+1/2}(U^n_{i,j+1}-U^n_{i,j})-a_{i,j-1/2}(U^n_{i,j}-U^n_{i,j-1})\\&+a_{i+1/2,j+1/2}(U^n_{i+1,j+1}-U^n_{i,j})-a_{i-1/2,j-1/2}(U^n_{i,j}-U^n_{i-1,j-1})\\&+a_{i-1/2,j+1/2}(U^n_{i-1,j+1}-U^n_{i,j})-a_{i+1/2,j-1/2}(U^n_{i,j}-U^n_{i+1,j-1})]\end{aligned} \tag{11}$$

with

$$\hat{U}_{i,j}=U^n_{i,j}-\frac{\Delta t}{h}[F_{i+1/2,j}-F_{i-1/2,j}]-\frac{\Delta t}{h}[G_{i,j+1/2}-G_{i,j-1/2}] \tag{12}$$

It is possible to write this in a more compact way, but to distinguish here between the filter and the basic step, we use this extended notation.

It is necessary to remark, that since the discrete filter is data dependent, it will not be monotone in every case. This is due to the fact, that we do not induce new numerical dissipation but distribute the existing diffusion in an optimal way. So there may situations occur, where it is not possible to make all coefficients of (8) positive. Due to this fact we consider this scheme as *quasi monotone*.

5. NUMERICAL RESULTS

In this section, we present numerical results of the developed schemes (11), (12). The test case we use is the initial value problem with fluxes $f(u)=0.5u^2$, $g(u)=u$ and entropy $\eta(u)=0.5u^2$

$$u(x,y,0)=\begin{cases}1.5, & x=0\\ -2.5x+1.5, & y=0\\ -1.0, & x=1\\ 0, & \text{else}\end{cases}$$

We compute the solution on a Cartesian grid with 50×50 points with CFL-number chosen to 1. The boundary condition on the upper side of the unit square are determined through simple extrapolation. The exact solution consist of constant regions to the left and the right, connected by a fan-like continuous wave which develops into a discontinuity (see Reference [12] for details). See Plates 1 and 2.

# cells	h	L^1-error	Order of accuracy
25	0.04	0.00541324	1.62
50	0.02	0.00191419	1.60
100	0.01	0.00067484	1.59

6. CONCLUSION

We have presented a data dependent discrete filter for oscillatory numerical schemes for conservation laws. The algorithm is not full monotone in the sense that the scheme is really positive, but it reduces the oscillation in the vicinity of discontinuities in a suitable manner. The constructed filter can be classified as an anisotropic diffusion filter known from image processing.

Further work will be necessary to clarify the question whether this algorithm can be extended to systems of conservation laws, namely the Euler equations.

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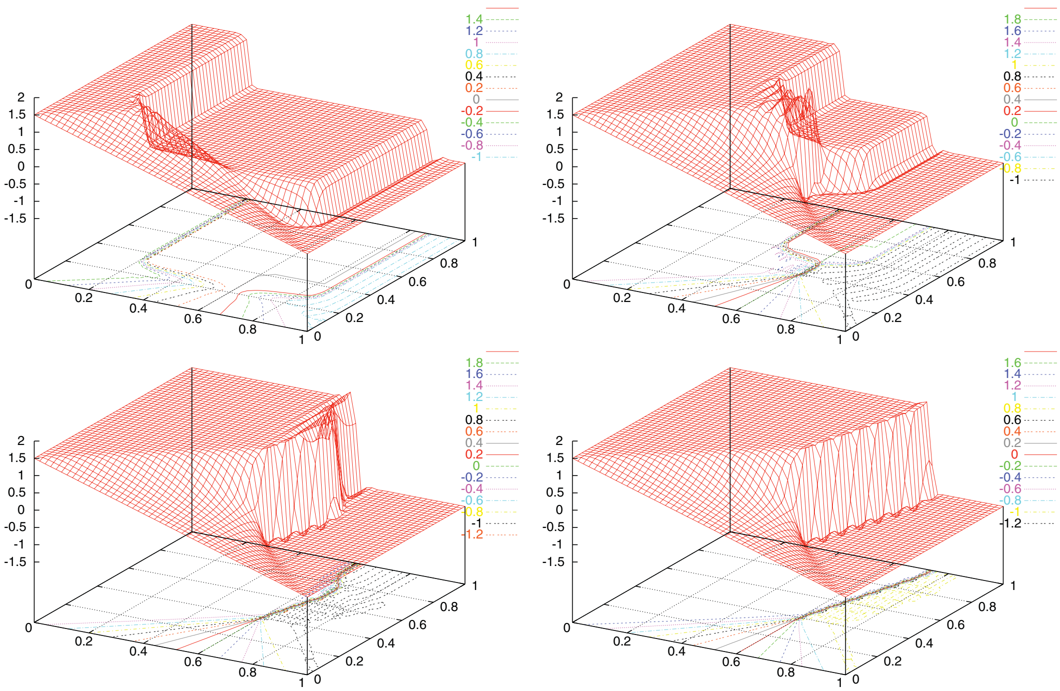

Plate 1. Filtered solution after 20, 40, 60 and 80 time steps.

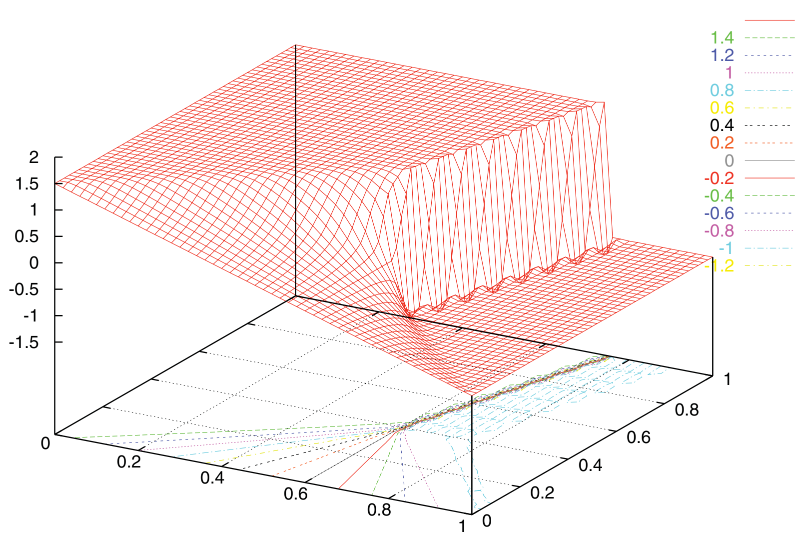

Plate 2. Steady-state solution of the test problem.